On the Cauchy-Euler Operator

Henrik Stenlund*
Visilab Signal Technologies Oy, Finland

November 19, 2015

Abstract

We present some results obtained for the Cauchy-Euler differential operator, especially for the exponential function of it. [1]

0.1 Keywords

Cauchy-Euler operator, differential operators, Cauchy problem.

0.2 Mathematical Classification

Mathematics Subject Classification 2010: 34A12, 34A05, 34L40

0.3

1 Introduction

Our short study focuses on a differential operator $x\partial x$ which is known as the Cauchy-Euler operator [1]. The operator is more generally defined as having a polynomial prefactor but here we concentrate on having the first order only.

Assuming $x \in R$ we present some useful features of the Cauchy-Euler differential operator. It is applied to functions whose derivatives exist and we also assume that the functions can be expanded as a Taylor's power series. The operator appears to have interesting properties displayed here, some of which are believed to be new. Here ∂_x represents the partial derivative operator. The results are valid for analytic functions over the complex plane as well.

2 The Cauchy-Euler Operator

We begin by presenting targets of progressively increasing complexity for the operator. These are proven by differentiation.

$$(x\partial_x)x = x \tag{1}$$

*The author is grateful to Visilab Signal Technologies for supporting this work.
[1]Visilab Report #2015-11

$$(x\partial_x)^m \cdot x = x \tag{2}$$

$$(x\partial_x)x^n = n \cdot x^n \tag{3}$$

Therefore we obtain

$$(x\partial_x)^m x^n = n^m \cdot x^n \tag{4}$$

For the general case, $m \in N$, n^m is the eigenvalue and x^n is the eigenfunction, $n \in R, n \neq 0$.

The partial derivative and Cauchy-Euler operator have helpful commutators which prove useful in handling the Cauchy-Euler operator in more complex cases.

$$[x, \partial_x] = -1 \tag{5}$$

$$[x\partial_x, \partial_x x] = 0 \tag{6}$$

We apply the m'th power operator to a more challenging function and get after expanding the exponential function to a Taylor's power series

$$(x\partial_x)^m e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n n^m}{n!} \tag{7}$$

The exponential differential operator is defined by its power series. That leads to

$$e^{\beta x \partial_x} x^j = \sum_{n=0}^{\infty} \frac{\beta^n (x\partial_x)^n}{n!} x^j = \sum_{n=0}^{\infty} \frac{\beta^n j^n x^j}{n!} \tag{8}$$

$$= e^{\beta j} x^j = (e^{\beta} x)^j \tag{9}$$

Here we have taken into use a parameter $\beta \in C, \beta \neq 0$. Without losing generality, we assume the function $A(x)$ has a Taylor's series around the origin. Thus, by using the results above, we can write down a rather impressive general expression

$$e^{\beta x \partial_x} A(x) = e^{\beta x \partial_x} \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = \sum_{n=0}^{\infty} \frac{a_n e^{\beta x \partial_x} x^n}{n!} \tag{10}$$

$$= \sum_{n=0}^{\infty} \frac{a_n (e^{\beta} x)^n}{n!} = A(xe^{\beta}) \tag{11}$$

We have as simple examples of application

$$cos(\beta x \partial_x) A(x) = \frac{1}{2}[A(xe^{i\beta}) + A(xe^{-i\beta})] \tag{12}$$

$$sin(\beta x \partial_x) A(x) = \frac{1}{2i}[A(xe^{i\beta}) - A(xe^{-i\beta})] \tag{13}$$

$$cosh(\beta x \partial_x) A(x) = \frac{1}{2}[A(xe^{\beta}) + A(xe^{-\beta})] \tag{14}$$

$$sinh(\beta x \partial_x) A(x) = \frac{1}{2}[A(xe^{\beta}) - A(xe^{-\beta})] \tag{15}$$

$$e^{\beta x \partial_x} e^{-\eta x} = e^{-\eta x e^{\beta}} \tag{16}$$

3 Discussion

The Cauchy-Euler operator has an interesting general property (11). It may prove useful while transforming differential equations and in solving various Cauchy problems with differential operators of this type.

References

[1] WIKIPEDIA: *https://en.wikipedia.org/wiki/Cauchy-Euler operator* **(2015)**