The Inversion Formula Applied to Some Examples

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Abstract

This work discusses a few applications of the recent inversion formula to various functions. The examples range from elementary to more involved. The purpose is to indicate the way of using the inversion formula in general. The target audience is among physicists, chemists, biologists and in other sciences with less inclination to formal mathematics. This article is subject to future modifications.[1 2]

0.1 Keywords

inversion of functions, Taylor series, reversion of series

0.2 Mathematical Classification

Mathematics Subject Classification 2010: 11A25, 40E99, 32H02

1 Introduction

1.1 Inversion Formula

The inversion of an analytic function $f(z)$ with a variable z was recently introduced [1]. Let us specify

$$u = f(z) \ u, z \in C \tag{1}$$

The formula is an infinite power series whose coefficients can be calculated in principle for all powers as long as there is no singularity at the point of focus z_0. The singularity would appear if the inverse of the derivative

$$[\frac{1}{\frac{df(z)}{dz}}]_{z_0} \tag{2}$$

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[1]Visilab Report #2011-01. Revision 1, 1-17-2011. This document is subject to change when new application examples are added.

at z_0 is not finite. This will take place if the derivative is zero at z_0. The inversion formula is the following. Let

$$u = f(z) \tag{3}$$

and f(z) be analytic over the interior of a circle

$$r_0 = |z - z_0| \tag{4}$$

Let the inverse function $g(u)$ be analytic over the interior of a circle R_0 at u_0

$$R_0 = |z - z_0| \tag{5}$$

Then the Taylor series

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(u-u_0)^n}{n!} [(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^{n-1} \frac{1}{\frac{df(z)}{dz}}]_{z_0} \tag{6}$$

represents the inverse function. The necessary, but not sufficient, condition for convergence is that the first derivative of $f(z)$ is nonzero at the point of focus z_0. Proof of this formula is presented in [1]. The formula is valid for analytic functions and for real-valued functions as well. It gives a formal inversion method even if the function is not analytic.

1.2 Algorithm or Method

The method of processing the inversion is the following. We need to calculate coefficients at each n for the Taylor power series. The irregular coefficient part is as below.

$$[(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^{n-1} \frac{1}{\frac{df(z)}{dz}}]_{z_0} \tag{7}$$

This is evaluated at z_0 by differentiating $(n-1)$ times and evaluating each term at z_0. The full Taylor series coefficient can then be written as

$$\frac{(u-u_0)^n}{n!} [(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^{n-1} \frac{1}{\frac{df(z)}{dz}}]_{z_0} \tag{8}$$

When a "sufficient" number of terms is available, one may sometimes be able to deduce a general trend for the coefficients. In complex cases, no apparent regular behavior is seen. Usually it is wise to calculate a few terms more, than at first sight seems necessary. Some less trivial functions may produce unexpected terms, like zeroes, seemingly irregularly and one should not jump to conclusions too quickly.

While calculating the coefficients, it may become obvious, what is actually the underlying trend. A pattern may repeat itself in the calculations and that is usually a sign of a trend or a sensible formula for the coefficients. It is very useful at this point to recall power series expansions of known functions. Such are the binomial series, exponential function, various expansions for the logarithm, trigonometric and hyperbolic functions etc. In complex cases, there seems to be no trend and one has to suffice with a set of beginning terms approximating the inverse function. Still, there may be groups of terms in the series which are identified leaving the rest unidentified. Sometimes also representing the function with a combination of other equivalent functions may lead to a more favorable process of inversion.

The convergence of the new series of the polynomial $(u - u_0)^n$ should be tested carefully. There is not much that can be said in general about it. The reader should refer to standard convergence tests.

It is interesting to note that analyticity is not essential for the working of the formula. The inversion formula offers a method for generating a formal expansion for the inverse. The real variable case is free from this fact and convergence is most important. Analyticity requirement for a complex-valued function is defined in Appendix A.

2 Application of the Inversion Formula

In the following we apply this formula to a few examples of varying complexity. Some of them are analytic.

2.1 Example 1

The simple function below is attempted to be inverted. We are using a general point of focus z_0 here without fixing it to any certain value.

$$u = f(z) = \frac{1}{z} \tag{9}$$

This function is not analytic at $z = 0$ and has a simple pole at $z = 0$ but is elsewhere analytic. The inverse of the derivative is

$$\frac{1}{\frac{df(z)}{dz}} = -z^2 \tag{10}$$

We are now able to calculate each term in sequence, starting from n=1. The singular points of this function are at infinity and should not create any problems while working at finite values.

$$(\frac{1}{\frac{df(z)}{dz}})_{z_0} = -z_0^2, \quad n = 1 \tag{11}$$

$$(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz} \frac{1}{\frac{df(z)}{dz}})_{z_0} = 2z_0^3, \quad n = 2 \tag{12}$$

$$((\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^2 \frac{1}{\frac{df(z)}{dz}})_{z_0} = -2 \cdot 3z_0^4, \quad n = 3 \tag{13}$$

$$((\frac{1}{\frac{df(z)}{dz}}\frac{d}{dz})^3\frac{1}{\frac{df(z)}{dz}})_{z_0} = 2\cdot 3\cdot 4z_0^5, \quad n=4 \tag{14}$$

$$((\frac{1}{\frac{df(z)}{dz}}\frac{d}{dz})^4\frac{1}{\frac{df(z)}{dz}})_{z_0} = -2\cdot 3\cdot 4\cdot 5z_0^6, \quad n=5 \tag{15}$$

Now it is obvious that the pattern is there and we can write the general formula for the inverted function as follows.

$$z = z_0 + \sum_{n=1}^{\infty}\frac{(-1)^n\cdot(u-u_0)^n\cdot z_0^{n+1}\cdot n!}{n!} \tag{16}$$

After a simplification we have

$$z = z_0\cdot\sum_{n=0}^{\infty}(-1)^n\cdot(u-u_0)^n\cdot z_0^n \tag{17}$$

We recognize this to be the binomial expansion and we can replace the sum with the source, obtaining.

$$z = \frac{z_0}{1+z_0\cdot(u-u_0)} \tag{18}$$

On the other hand, we know that

$$u_0 = \frac{1}{z_0} \tag{19}$$

and we can place it to the result yielding

$$z = \frac{1}{u} \tag{20}$$

According to common knowledge, this is a correct inverse of the original function. The range of validity is with finite values of z delivered by finite values of u. It is worth noting that there is no dependence on the point of focus z_0 in the result, at the same time appreciating the fact that it must not be zero for us to operate with finite values of u_0

2.2 Example 2

The following function below is attempted to be inverted. We are using a general point of focus z_0 here without fixing it to any certain value.

$$u = f(z) = ln(z) \tag{21}$$

The inverse of the derivative is

$$\frac{1}{\frac{df(z)}{dz}} = z \tag{22}$$

We are now able to calculate each term in sequence, starting from n=1.

$$(\frac{1}{\frac{df(z)}{dz}})_{z_0} = z_0, \quad n = 1 \tag{23}$$

$$(\frac{1}{\frac{df(z)}{dz}}\frac{d}{dz}\frac{1}{\frac{df(z)}{dz}})_{z_0} = z_0, \quad n = 2 \tag{24}$$

$$((\frac{1}{\frac{df(z)}{dz}}\frac{d}{dz})^2\frac{1}{\frac{df(z)}{dz}})_{z_0} = z_0, \quad n = 3 \tag{25}$$

$$((\frac{1}{\frac{df(z)}{dz}}\frac{d}{dz})^3\frac{1}{\frac{df(z)}{dz}})_{z_0} = z_0, \quad n = 4 \tag{26}$$

$$((\frac{1}{\frac{df(z)}{dz}}\frac{d}{dz})^4\frac{1}{\frac{df(z)}{dz}})_{z_0} = z_0, \quad n = 5 \tag{27}$$

It is again obvious what is the pattern and we can write the general formula for the inverted function as follows.

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(u-u_0)^n}{n!} \cdot z_0 \tag{28}$$

After a simplification we have

$$z = z_0 \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \tag{29}$$

We recognize this to be the expansion of the exponential function.

$$z = z_0 \cdot exp(u - u_0) \tag{30}$$

Since

$$u_0 = ln(z_0) \tag{31}$$

we can place it to the equation and obtain

$$z = z_0 \cdot exp(u) \cdot exp(-ln(z_0)) \tag{32}$$

and after simplifying we get

$$z = z_0 \cdot exp(u)\frac{1}{z_0} = exp(u) \tag{33}$$

This is a known inverse function for the $ln(z)$. Notice that both functions $ln(z)$ and $exp(z)$ have branches in the complex case. In addition, the function $ln(z)$ was not analytic but we still ended up with an analytic inverse for it. For real variables, this is not an issue. In the real variable case, the limitation

$$0 < z \tag{34}$$

is a safe working range. It is worth noting again that there is no dependence on the point of focus z_0 in the end result.

2.3 Example 3

The following function is to be inverted. We are using a point of focus $z_0 = \pi/2$.

$$u = f(z) = sin(z) \cdot exp(-z) \tag{35}$$

The function is analytic at finite z. The inverse of the derivative is

$$\frac{1}{\frac{df(z)}{dz}} = \frac{-exp(z)}{sin(z) - cos(z)} \tag{36}$$

At the point of focus it becomes

$$[\frac{1}{\frac{df(z)}{dz}}]_{z_0} = -exp(\pi/2) \quad n = 1 \tag{37}$$

We start calculating terms, from n=1.

$$(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz} \frac{1}{\frac{df(z)}{dz}})_{z_0} = 0, \quad n = 2 \tag{38}$$

$$((\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^2 \frac{1}{\frac{df(z)}{dz}})_{z_0} = -2 \cdot exp(3 \cdot \pi/2), \quad n = 3 \tag{39}$$

$$((\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^3 \frac{1}{\frac{df(z)}{dz}})_{z_0} = -4 \cdot exp(4 \cdot \pi/2), \quad n = 4 \tag{40}$$

There seems not to be any pattern, at least as seen from this small number of terms. We need to approximate the inverted function with a polynomial.

$$z = z_0 + (u - u_0) \cdot (-exp(\pi/2)) + \frac{(u - u_0)^2 \cdot 0}{2!} + \dots \tag{41}$$

$$\dots + \frac{(u - u_0)^3 \cdot (-2 \cdot exp(3 \cdot \pi/2))}{3!} + \dots \tag{42}$$

$$\dots + \frac{(u - u_0)^4 \cdot (-4 \cdot exp(4 \cdot \pi/2))}{4!} \tag{43}$$

After a simplification we have

$$z = \frac{\pi}{2} + \frac{7}{6} - \frac{4 \cdot u}{3} \cdot exp(\pi/2) + \frac{1}{3} \cdot u^3 \cdot exp(3 \cdot \pi/2) - \frac{u^4}{6} \cdot exp(4 \cdot \pi/2) \tag{44}$$

The value at the point of focus is

$$u_0 = exp(-\pi/2) \approx 0.208 \tag{45}$$

We seem to have a converging series of u since the series is probably alternating and the coefficients resemble inverse factorials of the $(power - 1)$ (very scientific, isn't it!). It is left for the reader to carry on calculating more terms to this series and be assured of the convergence. In this case there is a dependence on the point of focus z_0 in the result implicitly.

2.4 Example 4

The following function is attempted to be inverted. We are using a general point of focus z_0 here without fixing it to any certain value.

$$u = f(z) = \frac{1}{1-z} \tag{46}$$

This function is not analytic at $z = 1$ and has a simple pole at $z = 1$ but is elsewhere analytic. The inverse of the derivative is

$$\frac{1}{\frac{df(z)}{dz}} = (1-z)^2 \tag{47}$$

We are now able to calculate each term in sequence, starting from n=1. The singular points of this function are at infinity and should not create any problems while working at finite values.

$$(\frac{1}{\frac{df(z)}{dz}})_{z_0} = (1-z_0)^2, \quad n = 1 \tag{48}$$

$$(\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz} \frac{1}{\frac{df(z)}{dz}})_{z_0} = -2 \cdot (1-z_0)^3, \quad n = 2 \tag{49}$$

$$((\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^2 \frac{1}{\frac{df(z)}{dz}})_{z_0} = 2 \cdot 3 \cdot (1-z_0)^4, \quad n = 3 \tag{50}$$

$$((\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^3 \frac{1}{\frac{df(z)}{dz}})_{z_0} = -2 \cdot 3 \cdot 4 \cdot (1-z_0)^5, \quad n = 4 \tag{51}$$

$$((\frac{1}{\frac{df(z)}{dz}} \frac{d}{dz})^4 \frac{1}{\frac{df(z)}{dz}})_{z_0} = 2 \cdot 3 \cdot 4 \cdot 5 \cdot (1-z_0)^6, \quad n = 5 \tag{52}$$

The pattern is there and we can write the general formula for the inverted function as follows.

$$z = z_0 - \sum_{n=1}^{\infty} \frac{(-1)^n \cdot (u-u_0)^n \cdot (1-z_0)^n \cdot n!}{n!} \tag{53}$$

After a simplification we have

$$z = z_0 + 1 - \sum_{n=0}^{\infty} (-1)^n \cdot (u-u_0)^n \cdot (1-z_0)^n \tag{54}$$

We again recognize this to be the binomial expansion and we can replace the sum with the source, obtaining.

$$z = z_0 + 1 - \frac{1}{1 + (1-z_0) \cdot (u-u_0)} \tag{55}$$

When simplified it becomes as follows.

$$z = \frac{z_0 + (1-z_0^2) \cdot (u-u_0)}{1 + (1-z_0) \cdot (u-u_0)} \tag{56}$$

This is the correct inverse of the original function as is easy to verify.

2.5 Example 5

The function below is inverted next. We are using a general point of focus z_0 here without fixing it to any certain value.

$$u = f(z) = \sqrt{z} \tag{57}$$

This function has a branch line as is well known. The inverse of the derivative is

$$\frac{1}{\frac{df(z)}{dz}} = 2 \cdot \sqrt{z} \tag{58}$$

We are now able to calculate each term in sequence, starting from n=1.

$$(\frac{1}{\frac{df(z)}{dz}})_{z_0} = 2, \quad n = 1 \tag{59}$$

All higher coefficients are zero. The pattern is trivial and we can write the general formula for the inverted function as follows.

$$z = z_0 + (u - u_0) \cdot 2 \cdot \sqrt{z_0} + \frac{(u - u_0)^2}{2!} \cdot 2 \tag{60}$$

Since

$$u_0 = \sqrt{z_0} \tag{61}$$

we can simplify to

$$z = u^2 \tag{62}$$

This is the correct inverse of the original function as is well known. Note again that the z_0-dependency disappeared and we also obtained a truly analytic function. As an exercise to the reader, please do the same in reverse to obtain the $\sqrt{z}$ function from the function u^2.

References

[1] Stenlund, Henrik:*Inversion Formula*, arXiv:**1008:0183v2[math.GM]** (2010)

[2] Arfken, Georg:*Mathematical Methods for Physicists*, Academic Press, II edition, London (1971), ISBN 0-12-059851-5

A Appendix. Analyticity Requirement for a Function

For a complex-valued function to be analytic in some region S of complex values, it must comply to the following [2]. The function can be divided to real and imaginary parts as follows.

$$f(z) = u(x, y) + i \cdot v(x, y) \tag{63}$$

The Cauchy-Riemann conditions must be obeyed, as below, throughout the region S.

$$\frac{du(x,y)}{dx} = \frac{dv(x,y)}{dy} \tag{64}$$

and

$$\frac{du(x,y)}{dy} = -\frac{dv(x,y)}{dx} \tag{65}$$

Also the derivatives must be continuous. If these requirements are valid in the whole complex plane (excluding infinities), the function is analytic everywhere and is called entire. The requirement of analyticity is severe for complex functions.